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Ultimate Strength Prediction of Continuous Fiber-Reinforced Brittle Matrix Composites

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A model to predict ultimate strength of continuous fiber-reinforced brittle matirix composites has been developed. A statistical theory for the strength of the uniaxially fiber-reinforced brittle matrix composite is presented. Material of matrix is assumed to be homogeneous and isotropic, so that the strength of material is anywhere constant, whilst that of fiber is considered to show Weibull statistical distribution. The theory may be utilized to optimize the biaxial and multidirectional tensile strength properties of laminated materials. The composite strength is estimated by assuming no interacting matrix cracks. The frictional shear stress caused by bridging fibers is involved in the strength computation. The predicted strength is compared to experimental results with LAS-Glass/Nicalon fiber composite.

Key Words: Brittle Matrix Composite, Weibull Statistics, Tensile Strength, Interfacial Friction, Fibers

Nomenclature

d_f : Fiber diameter
E : Elastic modulus
$G_{\delta z_i}(\sigma_i)$: Probability that the element δz_i fractures subjected to the stress less than σ_i
H : Fiber specimen length
$\langle h \rangle$: Average fracture distance
l : Slip zone length
m : Weibull modulus
P : Probability of failure
S_0 : Strength scaling parameter; fiber mean strength ($m=\infty$)
v : Volume fraction
α : Dimension factor
α_0 : Normalizing parameter
χ : Parameter defined by $4\tau/d_f$
σ : Normal stress
σ_h : Fiber specimen failure stress
σ_i : Fiber stress at a distance z_i
σ_{ULT} : Ultimate strength
τ : Interfacial shear stress
ξ : Parameter defined by σ_m/S_0

Subscripts

c : Composite
f : Fiber
m : Matrix

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1. Introduction

In studies of composite materials, one important subject is to understand the effects of mechanical properties of constituents on composite strength. On the basis of understanding the mechanism, we could estimate a proper composite for a special use and predesign the constituent properties for a desired composite manufacturing.

Continuous fibers and whiskers have been commonly utilized to reinforce the materials. Advanced fibers show strong brittleness. It is well known that the strength of brittle materials is widely distributed compared to the conventional structural materials. Although one of the important characteristics specially in fibrous brittle composites such as ceramic matrix composites and intermetallic matrix composites is to obtain tougher material, only limited number of papers on the strength prediction of composite materials can be found (Rosen, 1964, Shu et. al., 1967, and

Sutcu, 1989). The major purpose in nonceramic matrix composites is to have the fiber bear greater proportion of the applied load. This load sharing depends on the ratio of fiber and matrix elastic moduli. In polymeric matrix composites, this ratio is very high, while in ceramic matrix composites, it is rather low and can be as low as unity. In the ceramic matrix composites, the purpose of reinforcing is mainly to enhance toughness by utilizing fiber/matrix interfacial characteristics. Thus, the mechanism of ultimate tensile strength should be understood on the basis of frictional resistance along the interface as well as fracture strength of constituents.

In this study, the interfacial friction force (Cho et. al., 1991, Holmes and Cho, 1992a, 1992b, Marshall, 1984) and statistical strength of fibers (Curtin, 1991) after matrix cracking occurs are considered to predict the ultimate tensile strength.

2. Composites Subjected to Uniaxial Tensile Loading

In the study, the brittle matrix is unidirectionally reinforced by continuous fibers as shown in Fig. 1. The matrix material is assumed to be homogeneous and isotropic, and the fiber is considered to be high strength brittle material. The fracture of the fiber depends on the size of surface flaw and its flaw distribution follows Weibull type statistics (Coleman, 1958). The fiber and matrix are considered to be frictionally bonded and/or weak chemical bonding.

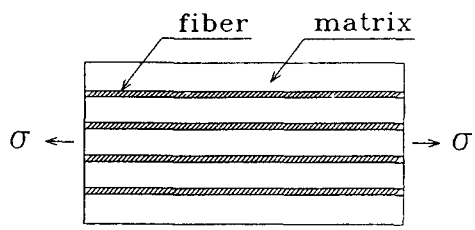


Fig. 1 (a) Fibrous composite model. The homogeneous and isotropic matrix material is uniaxially reinforced by high strength fibers

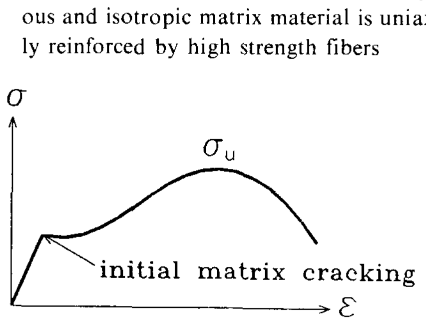


Fig. 1 (b) Monotonic tensile behavior of a fibrous composite is schematically shown.

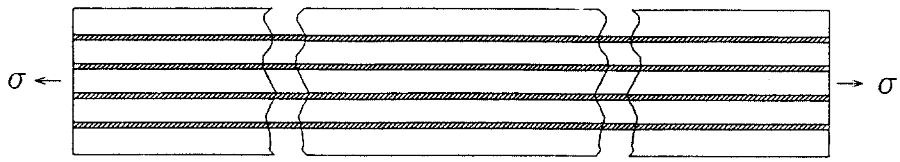
When the composite bears uniaxial tensile load, the average composite stress (σ_c) increases linearly with the composite strain until the first matrix crack forms. If the composite strain exceeds the matrix fracture strain, there occurs matrix cracking. Thus, the composite strain increases for a while under the constant tensile stress. Stress level at the matrix crack surface should be zero and load transferring from the intact fibers to the matrix through interfaces causes the matrix stress recovery in a distance depending on interfacial properties and external load. So the recovered matrix stress can make another matrix crack and the scenario repeats until the matrix stress is anywhere below the matrix strength (Cho et. al., 1992).

Figure 2 shows multiple matrix cracking phenomenon formed during tension tests. Since the matrix can bear no load at the crack, the fibers in the wake of the matrix cracking should bear greater load than prior to the formation of the matrix cracks. On the end of matrix cracking, the fibers break at some locations in a random fashion. And thus survived fibers should bear much higher proportion of the applied load as a result of fiber breaks. Finally, the fibers that bridge the matrix crack surfaces may attain to their ultimate stength and thus catastrophic failure occurs in the composite specimen.

3. Analysis of Composite Model

3.1 Application of weibull statistics

The tensile strength of the advanced fibers (B, C,SiC,Al_2O_3 etc) shows the statistical distribution. The failure of materials depends on flaw distribution and size. Weibull function is commonly used for the statistics of brittle materials (Coleman, 1958, Trustrum et. al., 1979, and Oh et. al., 1970). In the brittle material under the uni-



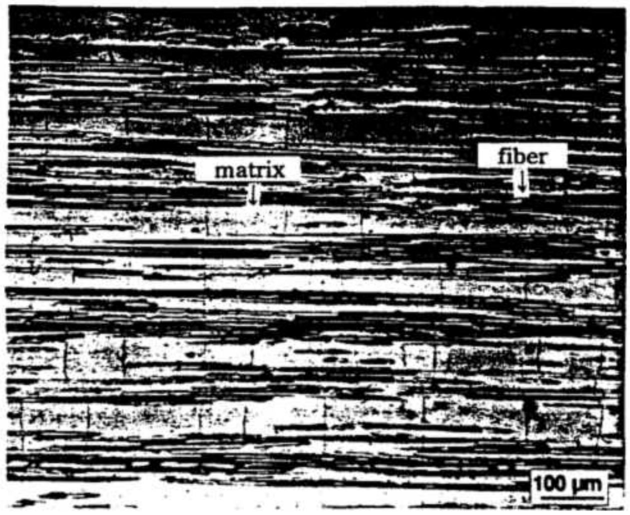


Fig. 2 Surface replica showing microcracking in a SiC/CAS composite after loading to 350MPa. The dimensionless mean crack spacings $\lambda \bar{l}$ is about 22 at the stress where $\lambda=2.2\times 10^4 m^{-1}$. The loading direction is from left to right

form load, the probability of failure at a stress σ is experessed by Weibull function.

$$P=1-\mathrm{Exp}\left\{-\frac{a}{a_0}\left(\frac{\sigma-\sigma_u}{\sigma_0}\right)^m\right\} \quad (1)$$

where m is the shape factor (Weibull modulus), a is a dimension (length, area, or volume) factor relating to the characteristics of flaws, and a_0 and σ_0 are normalizing parameters. And the material has no probability to fail at a stress less than σ_u. $a_0\sigma_0^m$ is a constant depending on the specimen. When a uniaxial tensile test is done on fibers, the tensile strength depends on the surface flaws and thus is a function of the surface area of the fiber. For the diameter d_f and length H of the fiber specimen, if a half of fibers fail up to σ_h the constant can be evaluated from Eq. (1).

$$a_0\sigma_0^m=\frac{\pi d_f H}{ln2}(\sigma_h-\sigma_u)^m \quad (2)$$

With experimental tensile testing data on reinforcing fibers, the material constant $a_0 a_m^m$ of the fiber can be estimated by using the derived equation above.

After the matrix cracking occurs, frictional slip zone along the interface of fiber and matrix forms in the matrix crack wake. The fiber stress which is a maximum on the matrix crack surface decreases at a rate of $4\tau/d_f$ along the interface and becomes a plane strain stress at the end of the slip zone where load transfer between fiber and matrix happens. The stress τ is defined as the shear stress working along the slip zone caused by the friction force.

Since the fracture of the fibers depends on the local stress, most of break locations are clustered in the vicinity of the matrix crack. In order to make calculation simpler, we may neglect fiber fractures outside the slip zone and utilize the slip zone length of Eq. (3).

$$l=\frac{d_f}{4\tau}\sigma=\frac{\sigma_m}{x} \quad (3)$$

wherer σ_m is the maximum fiber stress. The slip zone length l is calculated by considering plane strain condition and the result is $l=\dfrac{d_f\sigma E_c}{4\tau E_m v_m}$ wherer E and v stand for elastic moduli and volume fractions. So χ in Eq. (3) can be defined.

For the general case of nonuniform specimen stress, Eq. (1) is replaced by an integral form.

$$P=1-\text{Exp}\left\{-\frac{1}{A_0}\iint\left(\frac{\sigma-\sigma_u}{\sigma_0}\right)^m dA\right\} \tag{4}$$

Since a fiber fracture depends on the surface flaw rather than the defects in the volume, a was considered as the area. Neglecting the lower fracture limit stress σ_u, integration of Eq. (4) results in Eq. (5).

$$P=1-\text{Exp}\left\{-\frac{\pi d_f^2\tau^m}{2A_0\sigma_0^m(m+1)}\left(\frac{4l}{d_f}\right)^{m+1}\right\} \tag{5}$$

The Weibull distribution predicts the failure probability of one specimen, but does not provide the failure sites in the specimen. The fiber stress of the slip length $2l$ (Fig. 3) is symmetric about the matrix crack plane.

Following Oh and Finnie (1970) the probability density function may be derived. A fiber is divided into $2N$ elements of $\delta z=l/N$ each located at z_i from the matrix crack plane. Let $G_{\delta zi}(\sigma_i)$ denote the probability that the element δz_i fractures when subjected to the stress less than σ_i. A

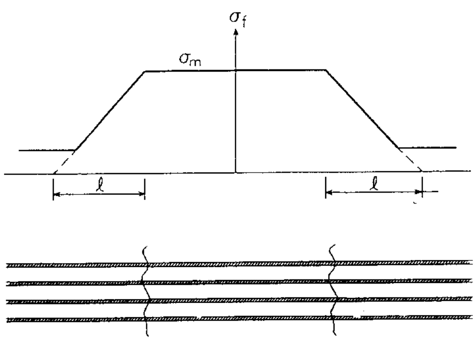


Fig. 3 Fibers are bridging over crack surfaces after matrix failure occurs. Slip zone ($2l$) and stress distribution on fiber (σ_f) are shown. Survived fibers bridging between cracked matrix surfaces bear all external load (σ_m). Some fiber load transfers to the matrix along the slip zone by friction force. At the end of the slip zone fiber and matrix share the plane strain condition load

local stress σ_i is a function of the reference stress σ_m (stress at the matrix crack plane) and distance z_i). Thus $G_{\delta zi}(\sigma_i)$ may be expressed by $G_{\delta zi}(\sigma_m, z_i)$. The probability that the element survives at the stress less than σ_m is $[1-G_{\delta zi}(\sigma_m, z_i)]$. The probability that all elements survive is then the product of individual survival probabilities: $\prod_{i=1}^{N}[1-G_{\delta zi}(\sigma_m, z_i)]$.

The probability that the element at z_i fractures when the reference stress is between σ_m and $\sigma_m+\delta\sigma_m$ in a condition that it has survived when the reference stress is less than σ_m, is

$$\frac{\partial G_{\delta zi}(\sigma_m, z_i)}{\partial\sigma_m}\delta\sigma_m[1-G_{\delta zi}(\sigma_m, z_i)]\prod_{i=1}^{N}[1-G_{\delta zi}(\sigma_m, z_i)] \tag{6}$$

Assume that the number of flaws is large enough for distribution but still small enough so that the flaws do not interact. Substituting Weibull distribution into $G(\sigma)$, the probability that the element at z_i fails when subjected to stress σ_i is given by

$$G_{\delta zi}(\sigma_m, z_i)=1-\text{Exp}\left\{-\frac{\pi d_f\delta z_i}{A_0}\left(\frac{\sigma_i}{\sigma_0}\right)^m\right\} \tag{7}$$

Substituting Eq. (7) into Eq. (6) and doing differentiation give the following

$$\frac{\pi d_f\delta z_i}{A_0}\frac{m}{\sigma_0}\left(\frac{\sigma_i}{\sigma_0}\right)^{m-1}\cdot \text{Exp}\left\{-2\sum_{i=1}^{N}\frac{\pi d_f\delta z_i}{A_0}\left(\frac{\sigma_i}{\sigma_0}\right)^m\right\}\delta\sigma_m \tag{8}$$

where the factor 2 is due to symmetry. As the number of elements increases ($N\rightarrow\infty$), the probability density function $\phi(\sigma_m, z)$ is derived.

$$\phi(\sigma_m, z)d\sigma_m dz=\frac{\pi d_f}{A_0}\frac{m\sigma^{m-1}}{\sigma_0^m}\cdot \text{Exp}\left\{-\frac{2\pi d_f}{A_0}\int_0^l\left(\frac{\sigma}{\sigma_0}\right)^m dz\right\}d\sigma_m dz \tag{9}$$

Integrating Eq. (9) with weighting factor z, the average fracture distance from the matrix crack plane is estimated by

$$\langle h\rangle=\int_0^{\sigma_m}2\int_0^l z\phi\, d\sigma dz \tag{10}$$

Substituting Eq. (9) into Eq. (10) and considering the local stress σ as $\chi(l-z)$, Eq. (11) is obtained.

$$\langle h\rangle=\frac{1}{\chi}\int_0^{\sigma_m}\left(\frac{\sigma}{S_0}\right)^{m+1}\cdot \mathrm{Exp}\left\{-\left(\frac{\sigma}{S_0}\right)^{m+1}\right\}d\sigma \tag{11a}$$

where S_0 is defined by

$$S_0=\left\{\frac{2A_0\sigma_0^m\tau(m+1)}{\pi d_f^2}\right)^{1/(m+1)} \tag{11b}$$

Defining a new variable $t=\left(\frac{\sigma}{S_0}\right)^{m+1}$ and substituting the variable into Eq. (11a) then the expected failure site of the fiber is expressed by

$$\langle h\rangle=\frac{S_0}{\chi(m+1)}\int_0^{t_m}t^{1/(m+1)}e^{-t}dt \tag{12}$$

where $t_m=\left(\frac{\sigma_m}{S_0}\right)^{m-1}$. In Eq. (12) the integral part is defined by the incomplete Gamma function (Luke, 1975) $\gamma\left(\frac{m+2}{m+1},\ t_m\right)$ and thus the mean value of the fiber fracture sites is expected by

$$\langle h\rangle=\frac{S_0}{\chi(m+1)}\gamma\left(\frac{m+2}{m+1},\ t_m\right) \tag{13}$$

As t_m goes to infinity, all of fibers would break and the incomplete Gamma function in Eq. (13) is converted to the Gamma function Γ. So the expected failure distance is

$$\langle h\rangle=\frac{S_0}{\chi(m+1)}\Gamma\left(\frac{m+2}{m+1}\right) \tag{14}$$

When the tensile strength of a material is deterministic, the probability density function of stength is Dirac delta ($m\rightarrow\infty$) and then $\langle h\rangle=0$ predicts that all fibers break at the matrix crack plane where the fiber stress has an extreme value (see Fig. 4). Also, integrating Eq. (9) with a weighting factor σ results in the corresponding solution for the mean strength of all fibers as the following

$$\langle\sigma\rangle=\frac{mS_0}{m+1}\Gamma\left(\frac{m+2}{m+1}\right) \tag{15}$$

As Weibull modulus goes to infinity, Eq. (15) predicts $\langle\sigma\rangle=S_0$.

3.2 Ultimate tensile strength analysis of brittle matrix composites

Once that the composite is under the uniaxial tensile load, the composite stress increases linearly with the corresponding fiber stress. When the composite strain exceeds to the fracture strain of the matrix at any applied load, multiple matrix cracks occur and thus intact fibers which bridge the matrix cracks should bear higher load. After the matrix cracking process ceases, additional external load directly transfers to the fiber. During the process of load increase, there gradually form statistical fiber breaks. The broken fibers which still bridge the crack planes by the friction force formed between matrix and fibers, resist to be pulled out of the matrix. Finally, when it is impossible for the small number of unbroken fibers and frictionally resisting fibers to endure the additional increase of external load, the catastrophic failure occurs in the composite specimen.

After the occurrence of the matrix cracking, the fiber failure probability at σ_m is determined as $P(\sigma_m)$ by Eq. (5) and a portion of fibers, $1-P(\sigma_m)$ still survives. Since the fiber broken at a distance h resists the friction force of $\pi d_f h\tau$ to pull out, the composite stress of fiber volume fraction v_f, can be estimated by

$$\sigma_c=\sigma_m v_f[1-P(\sigma_m)]+\chi v_f\langle h\rangle P(\sigma_m) \tag{16}$$

The load needed to pull out the broken fiber is proportional to the failure distance and thus the expected friction force for all of the failed fibers is expressed in terms of the average failure distance.

Since the composite starts to fail castastrophically when the bridging fibers can not endure

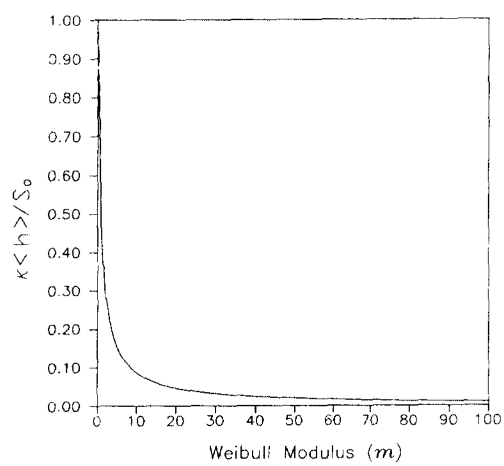


Fig. 4 Dimensionless mean break distance $\langle h\rangle$ of a fiber is shown with the change of Weibull modulus (m)

additional increase of the external load, in this stage the maximum fiber stress σ_m increases in a unstable fashion. And thus the condition for the catastrophic failure of the composite specimen can be expressed by

$$\frac{d\sigma_c}{d\sigma_m}=0 \tag{17}$$

Substitution of Eqs. (5), (16) and (13) into the instability condition Eq. (17) and defining a paramenter ξ by σ_m/S_0 may result in a nonlinear equation.

$$1-(m+2)\xi^{m+1}-\xi^{m+1}\mathrm{Exp}[-\xi^{m+1}] \\ +\xi^m\gamma\left(\frac{m+2}{m+1},\ \xi^{m+1}\right)=0 \tag{18}$$

Figure 5 shows the solution behavior with the increase of Weibull modulus. Let a solution of Eq. (18) be ξ_r and then $\sigma_m=\xi_r S_0$. Substituting Eqs. (5) and (13) into Eq. (16) and rearranging, the final equation to determine the ultimate tensile strength of the composite system is given by

$$\frac{\sigma_{ULT}}{v_f S_0}=\frac{1}{m+1}\gamma\left(\frac{m+2}{m+1},\ \xi_r^{m+1}\right) \\ +\left\{\xi_r-\frac{1}{m+1}\gamma\left(\frac{m+2}{m+1},\ \xi_r^{m+1}\right)\right\}\cdot \\ \mathrm{Exp}[-\xi_r^{m+1}] \tag{19}$$

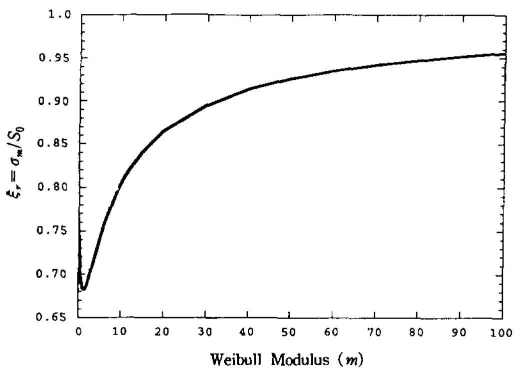


Fig. 5 The root behavior of Eq. (18) is shown against Weibull modulus

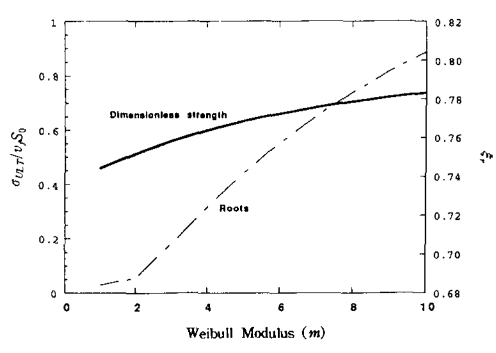


Fig. 6 Normalized ultimate tensile strength is plotted with Weibull modulus (see Eq. (19))

Figure 6 shows $\frac{\sigma_{ULT}}{v_f S_0}$ with Weibull modulus m. It may be deduced that the dimensionless tensile strength decreases with the increase of strength distribution.

In this model the fiber fractures are assumed to occur after matrix cracking and thus the predicted tensile strength of Eq. (19) should exceed the composite stress that corresponds to the matrix cracking stress σ_{mu} for validity. In the undamaged composite subjected to the tensile load, the strains of the composite, fiber and matrix are the same($\varepsilon_c=\varepsilon_f=\varepsilon_m$) until the first matrix crack occurs. When the composite strain becomes equal to the failure strain of the matrix, the composite stress is $\sigma_c=\frac{E_c\sigma_{mu}}{E_m}=\sigma_{mu}\left\{1+v_f\left(\frac{E_f}{E_m}-1\right)\right\}$. And thus in any case, this is the minimum tensile strength.

4. Numerical Example

The composite system of LAS glass-ceramic/Nicalon fiber was tested under uniaxial tension (Mah et. al., 1985, Sutcu, 1989 and Budiansky et. al., 1986). Using the data (see Table 1) in the references, $S_0=2190$MPa is evaluated by Eq. (11b). From Eq. (18) with $m=10$ (Sutcu, 1989) and the interfacial shear stress $\tau=2$MPa (Budiansky et. al., 1986), the root is $\xi_r=0.734759$(see Fig. 5) and the fiber volume fraction is given as $v_f=0.4$. Substitution of the constants into Eq. (19) predicts the ultimate tensile strength $\sigma_{ULT}=640$MPa. The experiment result (Mah et. al., 1985) was 570 MPa. For accurate prediction the experimental values of σ_h and m are very important. Figure 7 shows how the parameter S_0 depends on Weibull modulus m and the interfacial shear stress τ · For generating the figure some data (SiC fiber) from Table 1 were used. The figure says that the parameter which heavily affects the composite strength, dramatically

Table 1 Physical properties of matrix and fiber

Properties	Lithium Aluminosilicate (LAS)	SiC Fibers
Strength, MPa	172	2068(as received)
Strength, MPa	-	1450(after processing)
Elastic Modulus, GPa	83	193
Volume Fraction	0.6	0.4
Failure Strain, %	0.21	1.1
Weibull Modulus	-	10
Diameter, m	-	14×10^{-7}

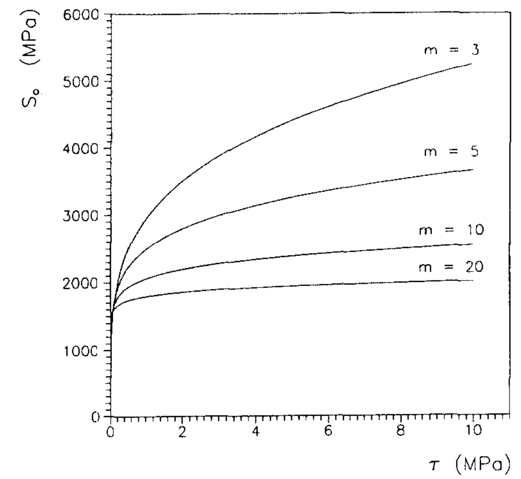


Fig. 7 Strength parameter S_0 is depicted against Weibull modulus m and the interfacial shear stress τ. Note that the parameter is very sensitive to the values of the Weibull modulus and the lower range of the shear stress

changes with the Weibull modulus and for the lower moduli the interfacial shear stress may have a great influence on the strength parameter.

5. Conclusions

The unidirectionally continuous fiber reinforced brittle matrix composites when subjected to an uniaxial tensile load are considered to analyze the tensile strength. The matrix is assumed to be homogeneous and isotropic elastic materials and the fiber has Weibull statistical characteristics for its tensile strength. The analysis in this paper provides new physical insights for the tensile strength of the brittle matrix composites. In particular, it shows how the interfacial friction force as well as mean fiber strength influences on the composite strength. The equation for predicting the tensile strength is also derived. The derived equation is very sensitive to the fiber strength data. Thus in order to use the equation, a caution should be placed on having the accurate data of the composite. The model of this research would give an upper limit of the strength.

Further efforts should be needed to obtain the closer prediction of tensile strength such as fiber debonding, Poisson effect, and fiber/matrix cracks interaction. Then the model may be refined.

Acknowledgement

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